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Wavelet Transformation and Wigner-Husimi Distribution Function

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Abstract We find that the optical wavelet transformation can be used to study the Husimi distribution function in phase space theory of quantum optics. We prove that the Husimi distribution function of a quantum state $|\psi\rangle$ is just the modulus square of the wavelet transform of $e^{-x^2/2}$ with $\psi(x)$ being the mother wavelet up to a Gaussian function. Thus a convenient approach for calculating various Husimi distribution functions of miscellaneous quantum states is presented.

Keywords Wavelet transformation · Wigner-Husimi distribution function · IWOP technique

Phase space technique has been proved very useful in various branches of physics. Distribution functions in phase space have been a major topic in studying quantum mechanics and quantum statistics. Among various phase space distributions the Wigner function $F_w(q, p)$ [1–10] is the most popularly used, since its two marginal distributions lead to measuring probability density in coordinate space and momentum space, respectively. But the Wigner distribution function itself is not a probability distribution due to being both positive and negative. In spite of its some attractive formal properties, it needs to be improved. To overcome this inconvenience, the Husimi distribution function $F_h(q', p')$ is introduced [11], which is defined in a manner that guarantees it to be nonnegative. Its definition is smoothing out the Wigner function by averaging over a "coarse graining" function,

$$F_h(q, p, \kappa) = 2\int\int_{-\infty}^{\infty} dq'dp' F_w(q', p') \exp\left[-\kappa(q' - q)^2 - \frac{(p' - p)^2}{\kappa}\right], \tag{1}$$

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where $\kappa > 0$ is the Gaussian spatial width parameter, which is free to be chosen and which determines the relative resolution in p-space versus q-space.

In this Letter we shall employ the optical wavelet transformation to study the Husimi distribution function. Wavelet is a 'small wave' localized in both time and frequency space [12–14]. This unique characteristic makes wavelets analysis in some ways superior to Fourier analysis which employs 'big waves', e.g., wavelets are particularly useful when processing data with sharp discontinuities or compressing image data. Mathematically, a wavelet ψ of the real variable x must satisfy the admissibility condition (AC) $\int_{-\infty}^{\infty} \psi(x)dx = 0$, which suggests that $\psi(x)$ behaves like a wave and decreases rapidly to zero as $|x|$ tends to infinity. Wavelet theory is concerned with the representation of a function in terms of a two-parameter family of dilates and translates of a fixed function, which is usually known as the "mother wavelet". A family of wavelets $\psi_{(\mu,s)}$ ($\mu > 0$, a scaling parameter; s, a translation parameter) are constructed from the mother wavelet ψ through

$$\psi_{(\mu,s)}(x) = \frac{1}{\sqrt{\mu}}\psi\left(\frac{x-s}{\mu}\right), \tag{2}$$

and the wavelet transform of a signal $f(x)$ by $\psi_{(\mu,s)}$ is defined by

$$W_\psi f(\mu, s) = \frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty} f(x)\psi^*\left(\frac{x-s}{\mu}\right)dx. \tag{3}$$

The AC ensures that the inverse transform and Parseval formula are applicable. When $\psi(x)$ is odd, it satisfies the AC obviously. A typical even function $\psi(x)$ is the Mexican hat wavelet $\psi(x) = \pi^{-1/4}e^{-x^2/2}(1-x^2)$, satisfying the AC.

In the following we shall show that the Husimi distribution function of a quantum state $|\psi\rangle$ can be obtained by making a wavelet transform of the Gaussian function $e^{-x^2/2}$, i.e.,

$$\langle\psi|\Delta_h(q, p, \kappa)|\psi\rangle = e^{-\frac{p^2}{\kappa}}\left|\int_{-\infty}^{\infty}\psi^*\left(\frac{x-s}{\mu}\right)e^{-x^2/2}dx\right|^2, \tag{4}$$

where

$$s = \frac{1}{\sqrt{\kappa}}(\kappa q + ip), \qquad \mu = \sqrt{\kappa}, \tag{5}$$

and $\langle\psi|\Delta_h(q, p, \kappa)|\psi\rangle$ is the Husimi distribution function as well as $\Delta_h(q, p, \kappa)$ is the Husimi operator,

$$\Delta_h(q, p) = \frac{2\sqrt{\kappa}}{1+\kappa} : \exp\left\{\frac{-\kappa(q-X)^2}{1+\kappa} - \frac{(p-P)^2}{1+\kappa}\right\} : , \tag{6}$$

here $: \ :$ denotes normal ordering; X and P are the coordinate and the momentum operators, related to the Bose operators $(a, a^\dagger)$, $[a, a^\dagger] = 1$ by $X = (a + a^\dagger)/\sqrt{2}$ and $P = (a - a^\dagger)/(\sqrt{2}i)$, respectively.

Proof of (4) According to Dirac's representation theory [15], we see that

$$\psi^*\left(\frac{x-s}{\mu}\right) = \left\langle\psi\middle|\frac{x-s}{\mu}\right\rangle, \qquad e^{-x^2/2} = \pi^{1/4}\langle x|0\rangle, \qquad |f\rangle = |0\rangle, \tag{7}$$

where ($|0\rangle$ is the vacuum state, $a|0\rangle = 0$) $|x\rangle$ is the eigen-vector of coordinate X, $X|x\rangle = x|x\rangle$,

$$|x\rangle = \pi^{-1/4}\exp\left(-\frac{1}{2}x^2 + \sqrt{2}xa^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle, \tag{8}$$

and $\langle\psi|$ is the state vector corresponding to the given mother wavelet, $|f\rangle = |0\rangle$ is the state to be transformed, so we can express (3) as

$$\int_{-\infty}^{\infty} dx\psi^*\left(\frac{x-s}{\mu}\right)e^{-x^2/2}dx = \left\langle\psi\right|\int_{-\infty}^{\infty} dx\left|\frac{x-s}{\mu}\right\rangle\langle x|0\rangle. \tag{9}$$

Let us define

$$U(\mu, s) \equiv \frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\left|\frac{x-s}{\mu}\right\rangle\langle x|dx, \tag{10}$$

is the squeezing-translating operator [16–18]. To combine wavelet transforms with transforms of quantum states more tightly and clearly, using the technique of integration within an ordered product (IWOP) [19–22] of operators, we can directly perform the integral in (10) [23]

$$U(\mu, s) = e^{\frac{-s^2}{2(1+\mu^2)}}\exp\left[-\frac{a^{\dagger 2}}{2}\tanh\lambda - \frac{sa^\dagger}{\sqrt{2}}\text{sech}\lambda\right]\exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\text{sech}\lambda\right]$$
$$\times\exp\left[\frac{a^2}{2}\tanh\lambda + \frac{sa}{\sqrt{2}}\text{sech}\lambda\right], \tag{11}$$

where $\mu = e^\lambda$, $\text{sech}\lambda = \frac{2\mu}{1+\mu^2}$, $\tanh\lambda = \frac{\mu^2-1}{\mu^2+1}$, and we have used the operator identity $e^{ga^\dagger a} = {:}\exp[(e^g - 1)a^\dagger a]{:}$. In particular, when $s = 0$, (11) reduces to the single-mode squeezing operator $U(\mu, 0) = \exp[\frac{\lambda}{2}(a^2 - a^{\dagger 2})]$. [24–26] From (11) it then follows that

$$U(\mu, s)|0\rangle = \text{sech}^{1/2}\lambda\exp\left[\frac{-s^2}{2(1+\mu^2)} - \frac{a^\dagger s}{\sqrt{2}}\text{sech}\lambda - \frac{a^{\dagger 2}}{2}\tanh\lambda\right]|0\rangle. \tag{12}$$

Substituting (5) and $\tanh\lambda = \frac{\kappa-1}{\kappa+1}$, $\cosh\lambda = \frac{1+\kappa}{2\sqrt{\kappa}}$ into (12) yields

$$\begin{aligned} &e^{-\frac{p^2}{2\kappa}+\frac{ipq}{\kappa+1}}U(\mu = \sqrt{\kappa}, s = \sqrt{\kappa}q + ip/\sqrt{\kappa})|0\rangle \\ &= \left(\frac{2\sqrt{\kappa}}{1+\kappa}\right)^{1/2}\exp\left\{\frac{-\kappa q^2}{2(1+\kappa)} - \frac{p^2}{2(1+\kappa)}\right. \\ &\quad\left. + \frac{\sqrt{2}a^\dagger}{1+\kappa}(\kappa q + ip) + \frac{1-\kappa}{2(1+\kappa)}a^{\dagger 2}\right\}|0\rangle \equiv |p, q\rangle_\kappa, \end{aligned} \tag{13}$$

then the wavelet transform of (9) can be further expressed as

$$e^{-\frac{p^2}{2\kappa}+\frac{ipq}{\kappa+1}}\int_{-\infty}^{\infty}\psi^*\left(\frac{x-(\sqrt{\kappa}q + ip/\sqrt{\kappa})}{\sqrt{\kappa}}\right)e^{-x^2/2}dx = \langle\psi|p, q\rangle_\kappa. \tag{14}$$

Using normally ordered form of the vacuum state projector $|0\rangle\langle 0| =: e^{-a^\dagger a}:$, and the IWOP method as well as (13) we have

$$\begin{aligned}
|p,q\rangle_{\kappa\kappa}\langle p,q| &= \frac{2\sqrt{\kappa}}{1+\kappa} : \exp\left[\frac{-\kappa}{1+\kappa}q^2 + \frac{1-\kappa}{2(1+\kappa)}(a^2+a^{\dagger 2}) - \frac{p^2}{1+\kappa}\right. \\
&\quad \left. + \frac{\kappa}{1+\kappa}\sqrt{2}q(a+a^\dagger) - \frac{1}{1+\kappa}i\sqrt{2}p(a-a^\dagger) - a^\dagger a\right]: \\
&= \frac{2\sqrt{\kappa}}{1+\kappa} : \exp\left[\frac{-\kappa}{1+\kappa}\left(q - \frac{a+a^\dagger}{\sqrt{2}}\right)^2 - \frac{1}{1+\kappa}\left(p - \frac{a-a^\dagger}{\sqrt{2}i}\right)^2\right]: \\
&= \frac{2\sqrt{\kappa}}{1+\kappa} : \exp\left[\frac{-\kappa}{1+\kappa}(q-X)^2 - \frac{1}{1+\kappa}(p-P)^2\right] : = \Delta_h(q,p,\kappa).
\end{aligned} \tag{15}$$

Now we explain why $\Delta_h(q,p,\kappa)$ is the Husimi operator. Using the formula for converting an operator A into its Weyl ordering form [27]

$$A = 2\int \frac{d^2\beta}{\pi}\langle -\beta|A|\beta\rangle \,\vdots\, \exp\{2(\beta^* a - a^\dagger\beta + a^\dagger a)\} \,\vdots\,, \qquad \beta = \beta_1 + i\beta_2, \tag{16}$$

where the symbol $\vdots\ \vdots$ denotes the Weyl ordering, $|\beta\rangle$ is the usual coherent state [28, 29], substituting (15) into (16) and performing the integration by virtue of the technique of integration within a Weyl ordered product of operators, we obtain

$$\begin{aligned}
|p,q\rangle_{\kappa\kappa}\langle p,q| &= \frac{4\sqrt{\kappa}}{1+\kappa}\int \frac{d^2\beta}{\pi}\langle -\beta| : \exp\left[-\frac{\kappa(q-X)^2}{1+\kappa} - \frac{(p-P)^2}{1+\kappa}\right] : |\beta\rangle \\
&\quad \times \,\vdots\, \exp\{2(\beta^* a - a^\dagger\beta + a^\dagger a)\} \,\vdots \\
&= \int \frac{4\sqrt{\kappa}\,d\beta_1 d\beta_2}{\pi(1+\kappa)} \,\vdots\, \exp\left[-2|\beta|^2 - \frac{\kappa(q-\sqrt{2}i\beta_2)^2}{1+\kappa} - \frac{(p+i\sqrt{2}\beta_1)^2}{1+\kappa}\right. \\
&\quad \left. + 2(\beta^* a - a^\dagger\beta + a^\dagger a)\right] \vdots \\
&= 2 \,\vdots\, \exp\left[\frac{1+\kappa}{2\kappa}\left(a - a^\dagger - \frac{\sqrt{2}ip}{1+\kappa}\right)^2 - \frac{1+\kappa}{2}\left(a + a^\dagger - \frac{\sqrt{2}\kappa q}{1+\kappa}\right)^2\right. \\
&\quad \left. + 2a^\dagger a - \frac{\kappa q^2 + p^2}{1+\kappa}\right] \vdots \\
&= 2 \,\vdots\, \exp\left[-\kappa(q-X)^2 - \frac{(p-P)^2}{\kappa}\right] \vdots\, .
\end{aligned} \tag{17}$$

This is the Weyl ordering form of $|p,q\rangle_{\kappa\kappa}\langle p,q|$. Then according to Weyl quantization scheme [30] we know the classical corresponding function of a Weyl ordered operator is obtained by just replacing $X \to q'$, $P \to p'$,

$$\vdots\, \exp\left[-\kappa(q-X)^2 - \frac{(p-P)^2}{\kappa}\right] \vdots \to \exp\left[-\kappa(q-q')^2 - \frac{(p-p')^2}{\kappa}\right], \tag{18}$$

and in this case the Weyl rule is expressed as

$$|p,q\rangle_{\kappa\kappa}\langle p,q| = 2\int dq'dp' \vdots \delta(q'-X)\delta(p'-P) \vdots \exp\left[-\kappa(q-q')^2 - \frac{(p-p')^2}{\kappa}\right]$$
$$= 2\int dq'dp'\Delta_w(q',p')\exp\left[-\kappa(q'-q)^2 - \frac{(p'-p)^2}{\kappa}\right], \tag{19}$$

where at the last step we used the Weyl ordering form of the Wigner operator $\Delta_w(q,p)$ [31]

$$\Delta_w(q,p) = \vdots \delta(q-X)\delta(p-P) \vdots. \tag{20}$$

In reference to (1) in which the relation between the Husimi function and the Wigner function is shown, we know that the right-hand side of (19) should be just the Husimi operator, i.e.

$$|p,q\rangle_{\kappa\kappa}\langle p,q| = 2\int dq'dp'\Delta_w(q',p')\exp\left[-\kappa(q'-q)^2 - \frac{(p'-p)^2}{\kappa}\right]$$
$$= \Delta_h(q,p,\kappa), \tag{21}$$

thus (4) is proved by combining (21) and (14). □

In summary, we have found that the optical wavelet transformation can be used to study the Husimi distribution function in quantum optics phase space theory. We have proved the Husimi distribution function of a quantum state $|\psi\rangle$ is just the wavelet transform of $e^{-x^2/2}$ with $\psi(x)$ being the mother wavelet, i.e. $e^{-\frac{p^2}{\kappa}}|\int_{-\infty}^{\infty}\psi^*(\frac{x-s}{\mu})e^{-x^2/2}dx|^2 = \langle\psi|\Delta_h(q,p,\kappa)|\psi\rangle$.

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Appendix

We can check (21) by the following way.

Using the normally ordered form of the Wigner operator [31]

$$\Delta_w(q,p) = \frac{1}{\pi} : e^{-(q-X)^2-(p-P)^2} :, \tag{22}$$

we can further perform the integration in (1) and see

$$\Delta_h(q,p,\kappa) = 2\int\int_{-\infty}^{\infty} \frac{dq'dp'}{\pi} : e^{-(q'-X)^2-(p'-P)^2} : \exp\left[-\kappa(q'-q)^2 - \frac{(p'-p)^2}{\kappa}\right]$$
$$= \frac{2\sqrt{\kappa}}{1+\kappa} : \exp\left\{\frac{-\kappa(q-X)^2}{1+\kappa} - \frac{(p-P)^2}{1+\kappa}\right\} : = (6) = |p,q\rangle_{\kappa\kappa}\langle p,q|, \tag{23}$$

which is the confirmation of (21).

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